Invariants of Lie algebras with fixed structure of nilradicals

2007 J. Phys. A: Math. Theor. 40 113

(http://iopscience.iop.org/1751-8121/40/1/006)

Institute of Physics Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **40** (2007) 113–130 doi:10.1088/1751-8113/40/1/006

Invariants of Lie algebras with fixed structure of nilradicals

Vyacheslav Boyko[1], **Jiri Patera**[2] **and Roman Popovych**[1,3]

[1] Institute of Mathematics of NAS of Ukraine, 3 Tereshchenkivs'ka Str., Kyiv-4, 01601, Ukraine
[2] Centre de Recherches Mathématiques, Université de Montréal, CP 6128 succursale Centre-ville, Montréal (Québec), H3C 3J7, Canada
[3] Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, A-1090 Wien, Austria

E-mail: boyko@imath.kiev.ua, rop@imath.kiev.ua and patera@CRM.UMontreal.CA

Received 5 October 2006
Published 6 December 2006
Online at stacks.iop.org/JPhysA/40/113

Abstract

An algebraic algorithm is developed for computation of invariants ('generalized Casimir operators') of general Lie algebras over the real or complex number field. Its main tools are the Cartan's method of moving frames and the knowledge of the group of inner automorphisms of each Lie algebra. Unlike the first application of the algorithm in Boyko *et al* (2006 *J. Phys. A: Math. Gen.* **39** 5749 (*Preprint* math-ph/0602046)), which deals with low-dimensional Lie algebras, here the effectiveness of the algorithm is demonstrated by its application to computation of invariants of solvable Lie algebras of general dimension $n < \infty$ restricted only by a required structure of the nilradical. Specifically, invariants are calculated here for families of real/complex solvable Lie algebras. These families contain, with only a few exceptions, all the solvable Lie algebras of specific dimensions, for whom the invariants are found in the literature.

PACS numbers: 02.20.−a, 02.20.Sv, 02.40.Vh
Mathematics Subject Classification: 17B05, 17B10, 17B30, 22E70, 58D19, 81R05

1. Introduction

The term Casimir operator was born in the physics literature about half a century ago as a reference to [6]. At that time only the lowest rank Lie algebras appeared of interest. In subsequent years the need to know the invariant operators of much larger Lie algebras grew more rapidly in physics than in mathematics.

In the mathematics literature it was soon recognized that the universal enveloping algebra $U(\mathfrak{g})$ of a semisimple Lie algebra $\mathfrak{g}$ contains elements that commute with $\mathfrak{g}$, that there is a basis for all such invariants and that the number of basis elements coincides with the rank of $\mathfrak{g}$. The degrees of the basis elements are given by the values of the exponents of the corresponding Weyl group (augmented by 1). The exponents are listed in many reference texts, see for example [3]. Best known are the Casimir operators of degree 2 for every semisimple Lie algebra. The actual form of a Casimir operators depends on the choice of basis of $\mathfrak{g}$.

Soon after the analogous question about the invariant operators was asked also for non-semisimple Lie algebras. An answer exhausting all cases appears out of reach at present. However methods applicable to specific Lie algebras were invented and used [17].

There are numerous papers on properties and specific computation of invariant operators of Lie algebras, on estimation of their number and on application of invariants of various classes of Lie algebras, or even a particular Lie algebra which appears in physical problems (see [1, 2, 4, 5, 13–15, 17, 19–22] and references therein).

The purpose of the paper is to present the latest version of the method first used for low-dimensional Lie algebras in [2], and to demonstrate its effectiveness by computing the invariants for families of solvable Lie algebras of general dimension. The families are distinguished by the structure of the nilradicals of their Lie algebras.

The main advantage of the method is in that it is purely algebraic. It eliminates the need to solve systems of differential equations of the conventional method, replacing them by algebraic equations. Moreover, efficient exploitation of the new method imposes certain constraints on the choice of bases of the Lie algebras. This then leads to simpler expressions for the invariants. In some cases the simplification is considerable.

Our paper is organized as follows.

After short review of necessary notions and results in section 2, we formulate the algebraic algorithm of the construction of the generalized Casimir operators of Lie algebras (section 3). It is based on the approach introduced in [2] for the case of algebras of arbitrary (fixed) dimension. The algorithm makes use of the Cartan's method of moving frames in the Fels–Olver version [7, 8]. More exactly, the notion of lifted invariants and different techniques of excluding parameters are applied.

In section 4, an illustrative example on invariants of a six-dimensional algebras is given for clear demonstration of features of the developed method. The main subject of our interest in the present paper is invariants, generalized Casimir operators, of solvable Lie algebras of arbitrary finite dimension $n < \infty$. For convenience all necessary notations are collected in section 5. A number of families of Lie algebras are considered further. The families are distinguished by the structure of their nilradicals. The invariant operators are found at once for all members of the family.

The Lie algebras with Abelian ideals of codimension 1 are completely investigated in the case of the both complex and real fields in section 6. The nilradicals of the algebras studied in section 7 are isomorphic to the simplest filiform algebras. Consideration of nilpotent algebras of strictly upper triangle matrices in section 8 is most sophisticated. At the same time, the developed method allows us to clarify an origin of the Casimir operators for these algebras, which was first found in [22].

All these examples illustrate various aspects and advantages of the proposed method.

2. Preliminaries

Consider a Lie algebra $\mathfrak{g}$ of dimension $\dim\mathfrak{g} = n < \infty$ over the complex or real field and the corresponding connected Lie group G. Let $\mathfrak{g}^*$ be the dual space of the vector space $\mathfrak{g}$. The map $\mathrm{Ad}^*\colon G \to GL(\mathfrak{g}^*)$ defined for any $g \in G$ by the relation

$$\langle \mathrm{Ad}^*_g x, u\rangle = \langle x, \mathrm{Ad}_{g^{-1}} u\rangle \quad \text{for all} \quad x \in \mathfrak{g}^* \quad \text{and} \quad u \in \mathfrak{g}$$

is called the *coadjoint representation* of the Lie group G. Here $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is the usual adjoint representation of G in $\mathfrak{g}$, and the image Ad_G of G under Ad is the inner automorphism group $\mathrm{Int}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The image of G under Ad^* is a subgroup of $GL(\mathfrak{g}^*)$ and is denoted by Ad^*_G.

A function $F \in C^\infty(\mathfrak{g}^*)$ is called an *invariant* of Ad^*_G if $F(\mathrm{Ad}^*_g x) = F(x)$ for all $g \in G$ and $x \in \mathfrak{g}^*$.

The set of invariants of Ad^*_G is denoted by $\mathrm{Inv}(\mathrm{Ad}^*_G)$. The maximal number $N_\mathfrak{g}$ of functionally independent invariants in $\mathrm{Inv}(\mathrm{Ad}^*_G)$ coincides with the codimension of the regular orbits of Ad^*_G, i.e. it is given by the difference

$$N_\mathfrak{g} = \dim\mathfrak{g} - \mathrm{rank}\,\mathrm{Ad}^*_G.$$

Here $\mathrm{rank}\,\mathrm{Ad}^*_G$ denotes the dimension of the regular orbits of Ad^*_G. It is a basis independent characteristic of the algebra $\mathfrak{g}$, the same as $\dim\mathfrak{g}$ and $N_\mathfrak{g}$. Sometimes $\mathrm{rank}\,\mathrm{Ad}^*_G$ is called as the rank of the Lie algebra $\mathfrak{g}$ or the Dixmier's invariant. (Let us note that the first name is more often used for other numerical characteristics of Lie algebras, which can differ from the above one [9].)

To calculate invariants explicitly, one should fix a basis of the algebra. Any (fixed) set of basis elements $e_1, \ldots, e_n$ of $\mathfrak{g}$ satisfies the commutation relations

$$[e_i, e_j] = c^k_{ij} e_k, \qquad i, j, k = 1, \ldots, n,$$

where c^k_{ij} are components of the tensor of structure constants of $\mathfrak{g}$ in the chosen basis.

Let $x \to \check{x} = (x_1, \ldots, x_n)$ be the coordinates in $\mathfrak{g}^*$ associated with the dual basis to the basis $e_1, \ldots, e_n$. Given any invariant $F(x_1, \ldots, x_n)$ of Ad^*_G, one finds the corresponding invariant of the Lie algebra $\mathfrak{g}$ as symmetrization, $\mathrm{Sym}\,F(e_1, \ldots, e_n)$, of F. It is often called a *generalized Casimir operator* of $\mathfrak{g}$. If F is a polynomial, $\mathrm{Sym}\,F(e_1, \ldots, e_n)$ is a usual *Casimir operator*, i.e. an element of the centre of the universal enveloping algebra of $\mathfrak{g}$. More precisely, the symmetrization operator Sym acts only on the monomials of the forms $e_{i_1} \cdots e_{i_r}$, where there are non-commuting elements among $e_{i_1}, \ldots, e_{i_r}$, and is defined by the formula

$$\mathrm{Sym}\big(e_{i_1} \cdots e_{i_r}\big) = \frac{1}{r!} \sum_{\sigma \in S_r} e_{i_{\sigma_1}} \cdots e_{i_{\sigma_r}},$$

where $i_1, \ldots, i_r$ take values from 1 to n, $r \in \mathbb{N}$, the symbol S_r denotes the permutation group of r elements. The set of invariants of $\mathfrak{g}$ is denoted by $\mathrm{Inv}(\mathfrak{g})$.

A set of functionally independent invariants $F^l(x_1, \ldots, x_n)$, $l = 1, \ldots, N_\mathfrak{g}$, forms *a functional basis* (*fundamental invariant*) of $\mathrm{Inv}(\mathrm{Ad}^*_G)$, i.e. any invariant $F(x_1, \ldots, x_n)$ can be uniquely presented as a function of $F^l(x_1, \ldots, x_n)$, $l = 1, \ldots, N_\mathfrak{g}$. Accordingly the set of $\mathrm{Sym}\,F^l(e_1, \ldots, e_n)$, $l = 1, \ldots, N_\mathfrak{g}$, is called a basis of $\mathrm{Inv}(\mathfrak{g})$.

If the Lie algebra $\mathfrak{g}$ is decomposable into the direct sum of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then the union of bases of $\mathrm{Inv}(\mathfrak{g}_1)$ and $\mathrm{Inv}(\mathfrak{g}_2)$ is a basis of $\mathrm{Inv}(\mathfrak{g})$. Therefore, for classification of invariants of Lie algebras from a given class it is really enough for ones to describe only invariants of the indecomposable algebras from this class.

Our task here is to determine the basis of the functionally independent invariants for Ad^*_G and then to transform these invariants to the invariants of the algebra $\mathfrak{g}$. Any other invariant of $\mathfrak{g}$ is a function of the independent ones.

Any invariant $F(x_1, \ldots, x_n)$ of Ad^*_G is a solution of the linear system of first-order partial differential equations

$$X_i F = 0, \qquad \text{i.e.} \quad c^k_{ij} x_k F_{x_j} = 0,$$

where $X_i = c^k_{ij} x_k \partial_{x_j}$ is the infinitesimal generator of the one-parameter group $\{\mathrm{Ad}^*_G(\exp \varepsilon e_i)\}$ corresponding to e_i. The mapping $e_i \to X_i$ gives a representation of the Lie algebra $\mathfrak{g}$. It is faithful iff the centre of $\mathfrak{g}$ consists of zero only. In the terms of structure constants for the fixed basis, the rank of coadjoint representation can be found by the formula

$$\operatorname{rank} \operatorname{Ad}_G^* = \sup_{\check{x} \in \mathbb{R}^n} \operatorname{rank}\left(c_{ij}^k x_k\right)_{i,j=1}^n.$$

The standard method of construction of generalized Casimir operators consists of integration of the above system of partial differential equations. It turns out to be rather cumbersome calculations, once the dimension of Lie algebra is not one of the lowest few. Alternative methods use matrix representations of Lie algebras. They are not much easier and are valid for a limited class of representations.

The algebraic method of computation of invariants of Lie algebras presented in this paper is simpler and generally valid. It extends to our problem the exploitation of the Cartan's method of moving frames [7, 8].

3. The algorithm

Let us recall some facts from [7, 8] and adapt them to the particular case of the coadjoint action of G on $\mathfrak{g}^*$. Let $\mathcal{G} = \operatorname{Ad}_G^* \times \mathfrak{g}^*$ denote the trivial left principal Ad_G^*-bundle over $\mathfrak{g}^*$. The right regularization $\widehat{R}$ of the coadjoint action of G on $\mathfrak{g}^*$ is the diagonal action of Ad_G^* on $\mathcal{G} = \operatorname{Ad}_G^* \times \mathfrak{g}^*$. It is provided by the maps

$$\widehat{R}_g(\operatorname{Ad}_h^*, x) = (\operatorname{Ad}_h^* \cdot \operatorname{Ad}_{g^{-1}}^*, \operatorname{Ad}_g^* x), \qquad g, h \in G, \qquad x \in \mathfrak{g}^*,$$

where the action on the bundle $\mathcal{G} = \operatorname{Ad}_G^* \times \mathfrak{g}^*$ is regular and free. We call $\widehat{R}_g$ the *lifted coadjoint action* of G. It projects back to the coadjoint action on $\mathfrak{g}^*$ via the Ad_G^*-equivariant projection $\pi_{\mathfrak{g}^*}\colon \mathcal{G} \to \mathfrak{g}^*$. Any *lifted invariant* of Ad_G^* is a (locally defined) smooth function from $\mathcal{G}$ to a manifold, which is invariant with respect to the lifted coadjoint action of G. The function $\mathcal{I}\colon \mathcal{G} \to \mathfrak{g}^*$ given by $\mathcal{I} = \mathcal{I}(\operatorname{Ad}_g^*, x) = \operatorname{Ad}_g^* x$ is the *fundamental lifted invariant* of Ad_G^*, i.e. $\mathcal{I}$ is a lifted invariant and any lifted invariant can be locally written as a function of $\mathcal{I}$. Using an arbitrary function $F(x)$ on $\mathfrak{g}^*$, we can produce the lifted invariant $F \circ \mathcal{I}$ of Ad_G^* by replacing x with $\mathcal{I} = \operatorname{Ad}_g^* x$ in the expression for F. Ordinary invariants are particular cases of lifted invariants, where one identifies any invariant formed as its composition with the standard projection $\pi_{\mathfrak{g}^*}$. Therefore, ordinary invariants are particular functional combinations of lifted ones that happen to be independent of the group parameters of Ad_G^*.

In view of the above consideration, the proposed algorithm for the construction of invariants of Lie algebra $\mathfrak{g}$ can be briefly formulated in the following four steps.

1. *Construction of generic matrix $B(\theta)$ of* Ad_G^*. It is calculated from the structure constants of the Lie algebra by exponentiation. $B(\theta)$ is the matrix of an inner automorphism of the Lie algebra $\mathfrak{g}$ in the given basis $e_1, \dots, e_n$, $\theta = (\theta_1, \dots, \theta_r)$ are group parameters (coordinates) of $\operatorname{Int}(\mathfrak{g})$, and

$$r = \dim \operatorname{Ad}_G^* = \dim \operatorname{Int}(\mathfrak{g}) = n - \dim \mathrm{Z}(\mathfrak{g}),$$

   $\mathrm{Z}(\mathfrak{g})$ is the centre of $\mathfrak{g}$.
2. *Fundamental lifted invariant.* The explicit form of the fundamental lifted invariant $\mathcal{I} = (\mathcal{I}_1, \dots, \mathcal{I}_n)$ of Ad_G^* in the chosen coordinates $(\theta, \check{x})$ in $\operatorname{Ad}_G^* \times \mathfrak{g}^*$ is

$$(\mathcal{I}_1, \dots, \mathcal{I}_n) = (x_1, \dots, x_n) \cdot B(\theta_1, \dots, \theta_r),$$

   or briefly $\mathcal{I} = \check{x} \cdot B(\theta)$.

3. *Elimination of parameters by normalization*. We find a non-singular submatrix

$$\frac{\partial\left(\mathcal{I}_{j_1},\ldots,\mathcal{I}_{j_\rho}\right)}{\partial\left(\theta_{k_1},\ldots,\theta_{k_\rho}\right)}$$

of the maximal dimension ρ in the Jacobian matrix $\partial\mathcal{I}/\partial\theta$ and solve the equations $\mathcal{I}_{j_1}=c_1,\ldots,\mathcal{I}_{j_\rho}=c_\rho$ with respect to $\theta_{k_1},\ldots,\theta_{k_\rho}$. Here the constants $c_1,\ldots,c_\rho$ are chosen to lie in the range of values of $\mathcal{I}_{j_1},\ldots,\mathcal{I}_{j_\rho}$. After substituting the found solutions into the other lifted invariants, we obtain $N_\mathfrak{g}=n-\rho$ usual invariants $F^l(x_1,\ldots,x_n)$.
4. *Symmetrization*. The functions $F^l(x_1,\ldots,x_n)$ which form a basis of $\mathrm{Inv}(\mathrm{Ad}^*_G)$ are symmetrized to $\mathrm{Sym}\,F^l(e_1,\ldots,e_n)$. It is the desired basis of $\mathrm{Inv}(\mathfrak{g})$.

Let us give some remarks on steps of the algorithm.

In the first step we usually use second canonical coordinates on $\mathrm{Int}(\mathfrak{g})$ as group parameters θ and present the matrix $B(\theta)$ in the form

$$B(\theta)=\prod_{i=1}^{r}\exp\left(\theta_i\,\hat{\mathrm{ad}}_{e_{n-r+i}}\right),$$

where $e_1,\ldots,e_{n-r}$ are assumed to form a basis of $Z(\mathfrak{g})$; ad_v denotes the adjoint representation of $v\in\mathfrak{g}$ in $GL(\mathfrak{g})$: $\mathrm{ad}_v w=[v,w]$ for all $w\in\mathfrak{g}$, and the matrix of ad_v in the basis $e_1,\ldots,e_n$ is denoted as $\hat{\mathrm{ad}}_v$. In particular, $\hat{\mathrm{ad}}_{e_i}=\left(c_{ij}^k\right)_{j,k=1}^n$. Often the parameters θ are additionally transformed in a light manner (signs, renumbering, re-denotation etc) for simplification of the final presentation of $B(\theta)$. It is also sometimes convenient for us to introduce 'virtual' group parameters corresponding to centre basis elements.

Since $B(\theta)$ is a general form of matrices from $\mathrm{Int}(\mathfrak{g})$, we should not adopt it in any way for the second step.

In fact, the third step of our algorithm can involve different techniques of elimination of parameters which are also based on using an explicit form of lifted invariants [2]. The applied normalization procedure [7, 8] can also be modified and be used in more involved way (see e.g. section 6.2).

Let us emphasize that the maximal dimension of a non-singular submatrix in the Jacobian matrix $\partial\mathcal{I}/\partial\theta$ coincides with the rank of coadjoint representation of $\mathfrak{g}$, i.e.

$$\mathrm{rank}\,\mathrm{Ad}^*_G=\rho=\max_{\check{x}\in\mathbb{R}^n}\max_{\theta\in\mathbb{R}^r}\mathrm{rank}\frac{\partial\mathcal{I}}{\partial\theta}.$$

It gives one more formula for calculation of the rank of coadjoint representation.

In conclusion let us underline that the search of invariants of Lie algebra $\mathfrak{g}$, which has been done by solving a linear system of first-order partial differential equations, is replaced here by the construction of the matrix $B(\theta)$ of inner automorphisms and by excluding the parameters θ from the fundamental lifted invariant $\mathcal{I}=\check{x}\cdot B(\theta)$ in some way.

4. Illustrative example

The six-dimensional solvable Lie algebra $\mathfrak{g}^a_{6.38}$ [12] with five-dimensional nilradical $\mathfrak{g}_{3.1}\oplus 2\mathfrak{g}_1$ has the following non-zero commutation relations

$$[e_4,e_5]=e_1,\qquad [e_1,e_6]=2ae_1,\qquad [e_2,e_6]=ae_2-e_3,\qquad [e_3,e_6]=e_2+ae_3,$$
$$[a_4,e_6]=e_2+ae_4-e_5,\qquad [e_5,e_6]=e_3+e_4+ae_5,\qquad a\in\mathbb{R}.$$

Here we have modified the basis to K-canonical form [11], i.e. now $\langle e_1,\ldots,e_i\rangle$ is an ideal of $\langle e_1,\ldots,e_i,e_{i+1}\rangle$ for any $i=1,2,3,4,5$. (See also [2] for discussion of role of K-canonical bases in the investigation of solvable Lie algebras.)

The matrices of the adjoint representation $\hat{\mathrm{ad}}_{e_i}$ of the basis elements e_1, e_2, e_3, e_4, e_5 and e_6 correspondingly have the form

$$\begin{pmatrix}0&0&0&0&0&2a\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&a\\0&0&0&0&0&-1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&a\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\\0&0&0&0&0&a\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{pmatrix},\quad \begin{pmatrix}0&0&0&-1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&a\\0&0&0&0&0&0\end{pmatrix},\quad \begin{pmatrix}-2a&0&0&0&0&0\\0&-a&-1&-1&0&0\\0&1&-a&0&-1&0\\0&0&0&-a&-1&0\\0&0&0&1&-a&0\\0&0&0&0&0&0\end{pmatrix}.$$

The inner automorphisms of $\mathfrak{g}^a_{6.38}$ are then described by the block triangular matrix

$$B(\theta) = \prod_{i=1}^{5} \exp\left(\theta_i \hat{\mathrm{ad}}_{e_i}\right) \cdot \exp\left(-\theta_6 \hat{\mathrm{ad}}_{e_6}\right)$$

$$= \begin{pmatrix}\varepsilon^2 & 0 & 0 & -\theta_5\varepsilon\varkappa - \theta_4\varepsilon\sigma & -\varepsilon\theta_5\sigma + \varepsilon\theta_4\varkappa & -\frac{1}{2}\theta_5^2 + a\theta_4\theta_5 - \frac{1}{2}\theta_4^2 + 2a\theta_1\\ 0 & \varepsilon\varkappa & \varepsilon\sigma & \theta_6\varepsilon\varkappa & \theta_6\varepsilon\sigma & \theta_4+\theta_3+a\theta_2\\ 0 & -\varepsilon\sigma & \varepsilon\varkappa & -\theta_6\varepsilon\sigma & \theta_6\varepsilon\varkappa & \theta_5+a\theta_3-\theta_2\\ 0&0&0&\varepsilon\varkappa&\varepsilon\sigma&\theta_5+a\theta_4\\ 0&0&0&-\varepsilon\sigma&\varepsilon\varkappa&a\theta_5-\theta_4\\ 0&0&0&0&0&1\end{pmatrix},$$

where $\varepsilon = \mathrm{e}^{a\theta_6}$, $\varkappa = \cos\theta_6$, $\sigma = \sin\theta_6$. Therefore, a functional basis of lifted invariants is formed by

$$\begin{aligned}
\mathcal{I}_1 &= \varepsilon^2 x_1,\\
\mathcal{I}_2 &= \varepsilon(\varkappa x_2 - \sigma x_3),\\
\mathcal{I}_3 &= \varepsilon(\sigma x_2 + \varkappa x_3),\\
\mathcal{I}_4 &= \varepsilon((-\theta_5\varkappa - \theta_4\sigma)x_1 + \theta_6\varkappa x_2 - \theta_6\sigma x_3 + \varkappa x_4 - \sigma x_5),\\
\mathcal{I}_5 &= \varepsilon((-\theta_5\sigma + \theta_4\varkappa)x_1 + \theta_6\sigma x_2 + \theta_6\varkappa x_3 + \sigma x_4 + \varkappa x_5),\\
\mathcal{I}_6 &= \left(-\tfrac{1}{2}\theta_5^2 + a\theta_4\theta_5 - \tfrac{1}{2}\theta_4^2 + 2a\theta_1\right)x_1 + (\theta_4+\theta_3+a\theta_2)x_2 + (\theta_5+a\theta_3-\theta_2)x_3\\
&\quad + (\theta_5 + a\theta_4)x_4 + (a\theta_5 - \theta_4)x_5 + x_6.
\end{aligned}$$

The algebra $\mathfrak{g}^a_{6.38}$ has two independent invariants. They can be easily found from first three lifted invariants by the normalization procedure. Further the cases $a = 0$ and $a \neq 0$ should be considered separately since there exists difference between them in the normalization procedure.

It is obvious in case $a = 0$ that e_1 generating the centre $Z\left(\mathfrak{g}^0_{6.38}\right)$ is one of the invariants. The second invariant is found via combining the lifted invariants $\mathcal{I}_2$ and $\mathcal{I}_3$: $\mathcal{I}_2^2 + \mathcal{I}_3^2 = x_2^2 + x_3^2$. Since the symmetrization procedure is trivial for this algebra we obtain the following set of polynomial invariants:

$$e_1, \qquad e_2^2 + e_3^2.$$

In case $a \neq 0$ we solve the equation $\mathcal{I}_1 = 1$ with respect to $\mathrm{e}^{2a\theta_6}$ and substitute the obtained expression $\mathrm{e}^{2a\theta_6} = 1/x_1$ into the combinations $\mathcal{I}_2^2 + \mathcal{I}_3^2$ and $\exp(-2a \arctan \mathcal{I}_3/\mathcal{I}_2)$. In view of trivial symmetrization we obtain the final basis of generalized Casimir invariants

$$\frac{e_2^2 + e_3^2}{e_1}, \qquad e_1 \exp\left(-2a \arctan \frac{e_3}{e_2}\right).$$

It is equivalent to the one constructed in [4], but it contains no complex numbers and is written in a more compact form.

5. Notations

Further we use the following notations: $\operatorname{diag}(\alpha_1, \ldots, \alpha_k)$ is the $k \times k$ diagonal matrix with the elements $\alpha_1, \ldots, \alpha_k$ on the diagonal; $E^k = \operatorname{diag}(1, \ldots, 1)$ is the $k \times k$ unity matrix; E^k_{ij} (for the fixed values i and j) denotes the $k \times k$ matrix $(\delta_{ii'}\delta_{jj'})$ with i' and j' running the numbers of rows and column correspondingly, i.e. the $k \times k$ matrix with the unit on the cross of the ith row and the jth column and the zero otherwise; J^k_λ is the Jordan block of dimension k and the eigenvalue λ:

$$\left[J^k_\lambda\right]_{ij} = \begin{cases} \lambda, & \text{if} \quad j = i, \\ 1, & \text{if} \quad j - i = 1, \qquad i, j = 1, \ldots, k, \\ 0, & \text{otherwise}, \end{cases}$$

i.e.

$$J^k_\lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & \lambda & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}, \quad \exp\left(\theta J^k_0\right) = \begin{pmatrix} 1 & \theta & \frac{1}{2!}\theta^2 & \frac{1}{3!}\theta^3 & \cdots & \frac{1}{(k-1)!}\theta^{k-1} \\ 0 & 1 & \theta & \frac{1}{2!}\theta^2 & \cdots & \frac{1}{(k-2)!}\theta^{k-2} \\ 0 & 0 & 1 & \theta & \cdots & \frac{1}{(k-3)!}\theta^{k-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \theta \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

(let us note that $J^k_\lambda = \lambda E^k + J^k_0$ and therefore $\exp\left(\theta J^k_\lambda\right) = \mathrm{e}^{\lambda\theta} \exp\left(\theta J^k_0\right)$); $R^r_{\mu\nu}$ is the real Jordan block of dimension $r = 2k$, $k \in \mathbb{N}$, which corresponds to the pair of two complex Jordan blocks J^k_λ and $J^k_{\lambda^*}$ with the complex conjugate eigenvalues λ and λ^*, where $\mu = \operatorname{Re} \lambda$, $\nu = \operatorname{Im} \lambda \neq 0$:

$$R^2_{\mu\nu} = \begin{pmatrix} \mu & \nu \\ -\nu & \mu \end{pmatrix}, \qquad R^{2k}_{\mu\nu} = \left.\begin{pmatrix} R^2_{\mu\nu} & E^2 & 0 & 0 & \cdots & 0 \\ 0 & R^2_{\mu\nu} & E^2 & 0 & \cdots & 0 \\ 0 & 0 & R^2_{\mu\nu} & E^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & E^2 \\ 0 & 0 & 0 & 0 & \cdots & R^2_{\mu\nu} \end{pmatrix}\right\} k \text{ blocks;}$$

$A_1 \oplus A_2$ is the direct sum $\left(\begin{smallmatrix} A_1 & 0 \\ 0 & A_2 \end{smallmatrix}\right)$ of the square matrices A_1 and A_2; $A_1 \overset{C}{+} A_2$ is the block triangular matrix $\left(\begin{smallmatrix} A_1 & C \\ 0 & A_2 \end{smallmatrix}\right)$, where $A_1 \in M_{k,k}$, $A_2 \in M_{l,l}$, $C \in M_{k,l}$.

Above 0 denotes the zero matrices of different dimensions.

6. Solvable algebras with Abelian ideals of codimension 1

Consider a Lie algebra $\mathfrak{g}$ of dimension n with the Abelian ideal I of dimension $n-1$ (cf [11]). Let us suppose that the ideal I is spanned on the basis elements $e_1, e_2, \ldots, e_{n-1}$. Then the algebra $\mathfrak{g}$ is completely determined by the $(n-1)\times(n-1)$ matrix $M=(m_{kl})$ of restriction of the adjoint action ad_{e_n} on the ideal I. The (possibly) non-zero commutation relations of $\mathfrak{g}$ have the form

$$[e_k, e_n] = \sum_{l=1}^{n-1} m_{lk} e_l, \qquad k = 1, \ldots, n-1.$$

Due to possibility of scaling e_n, the matrix M and, therefore, its eigenvalues are determined up to multiplication on a non-zero number from the field under consideration. The matrix M is reduced to the Jordan canonical form by change of the basis in I:

$$M = J_{\lambda_1}^{r_1} \oplus \cdots \oplus J_{\lambda_s}^{r_s},$$

where $r_1+\cdots+r_s = n-1$, $r_i \in \mathbb{N}$, $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, s$. In the real case the direct sum of two complex blocks $J_{\lambda_i}^{r_i}$ and $J_{\lambda_j}^{r_j}$, where $r_i = r_j$ and λ_i is conjugate of λ_j, is assumed as replaced by the corresponding real Jordan block $R_{\mu\nu}^{2r_i}$ with $\mu = \mathrm{Re}\,\lambda_i$ and $\nu = \mathrm{Im}\,\lambda_i \neq 0$. The Jordan canonical form is unique up to permutation of the Jordan blocks.

The above algebra will be denoted as $\mathfrak{J}_{\lambda_1\ldots\lambda_s}^{r_1\ldots r_s}$. It is additionally assumed that $\mathfrak{J}_{\lambda'_1\ldots\lambda'_{s'}}^{r'_1\ldots r'_{s'}}$ denotes the same algebra if $s = s'$ and there exists a non-zero constant $\varkappa$ that $(\lambda'_i, r'_i) = (\varkappa\lambda_i, r_i)$, $i = 1, \ldots, s$, up to permutation of pairs (λ_i, r_i).

The Lie algebra $\mathfrak{J}_{\lambda_1\ldots\lambda_s}^{r_1\ldots r_s}$ is decomposable iff there exists a value of i such that $(\lambda_i, r_i) = (0, 1)$. (Then e_i is an invariant of $\mathfrak{J}_{\lambda_1\ldots\lambda_s}^{r_1\ldots r_s}$.) Hence the contrary condition is supposed to be satisfied below. It should be also noted this algebra is nilpotent iff $\lambda_1 = \cdots = \lambda_s = 0$.

6.1. Simplest cases

Consider the simplest case for M to be a single Jordan block with the eigenvalue λ, i.e. $\mathfrak{g} = \mathfrak{J}_\lambda^{n-1}$, $n = 2, 4, \ldots$. The value of λ can be normalized to 1 in case $\lambda \neq 0$ but it is convenient for the further consideration to avoid normalization of λ some time.

The non-zero commutation relations of $\mathfrak{J}_\lambda^{n-1}$ at most are

$$[e_1, e_n] = \lambda e_1, \qquad [e_k, e_n] = \lambda e_k + e_{k-1}, \qquad k = 2, \ldots, n-1, \qquad \lambda \in \mathbb{C}.$$

(The first one is zero if $\lambda = 0$.) Therefore, its inner automorphisms are described by the triangular matrix

$$B(\theta) = \exp\left(\theta_n J_\lambda^{n-1}\right) \overset{C}{+} E^1, \qquad C = (\theta_2+\lambda\theta_1, \theta_3+\lambda\theta_2, \ldots, \theta_{n-1}+\lambda\theta_{n-2}, \lambda\theta_{n-1})^{\mathrm{T}},$$

i.e. a functional basis of lifted invariants is formed by

$$\widehat{\mathcal{I}}_k = \mathrm{e}^{\lambda\theta_n}\mathcal{I}_k, \qquad k = 1, \ldots, n-1, \qquad \widehat{\mathcal{I}}_n = \mathcal{I}_n + \lambda\sum_{j=1}^{n-1}\theta_j x_j,$$

where

$$\mathcal{I}_k = \sum_{j=1}^{k} \frac{\theta_n^{k-j}}{(k-j)!} x_j, \qquad k = 1, \ldots, n-1, \qquad \mathcal{I}_n = \sum_{j=1}^{n-2}\theta_{j+1}x_j + x_n. \tag{1}$$

The nilpotent ($\lambda = 0$) and solvable ($\lambda \neq 0$) cases of $\mathfrak{J}_\lambda^{n-1}$ should be considered further separately since there exists difference in the normalization procedure. The dimension $n = 2$ is singular in both the cases. $\mathfrak{J}_0^1$ is the two-dimensional Abelian Lie algebra and therefore has two independent invariants, namely e_1 and e_2. $\mathfrak{J}_1^1$ is the two-dimensional non-Abelian Lie algebra and therefore has no invariants. We assume below that $n \geqslant 3$.

The algebra $\mathfrak{J}_0^{n-1}$ is, in some sense, the simplest filiform algebra of dimension n. Let us note that the adjoint representation of $\mathfrak{J}_0^{n-1}$ is unfaithful since the centre $Z(\mathfrak{J}_0^{n-1}) = \langle e_1 \rangle \neq \{0\}$. Therefore, there are $n-1$ parameters in the expression of $B(\theta)$ excluding θ_1, and $\widehat{\mathcal{I}}$ coincides with $\mathcal{I}$. It is obvious that the element e_1 generating $Z(\mathfrak{J}_0^{n-1})$ is one of the invariants, which corresponds to $\mathcal{I}_1 = x_1$. Another $(n-3)$ invariants are found by the normalization procedure applied to the lifted invariants $\mathcal{I}_2, \ldots, \mathcal{I}_{n-1}$. Namely, we solve the equation $\mathcal{I}_2 = 0$ with respect to θ_n and then substitute the obtained expression $\theta_n = -x_2/x_1$ into the other $\mathcal{I}$'s. To construct polynomial invariants finally, we multiply the derived invariants by powers of the invariant x_1. Since the symmetrization procedure is trivial for this algebra, we get the following complete set of independent generalized Casimir operators which are classical (i.e. polynomial) Casimir operators:

$$\xi_1 = e_1, \qquad \xi_k = \sum_{j=1}^{k} \frac{(-1)^{k-j}}{(k-j)!} e_1^{j-2} e_2^{k-j} e_j, \qquad k = 3, \ldots, n-1. \tag{2}$$

This set was first constructed by the moving frame approach in example 6 of [2] and completely coincides with the one determined in lemma 1 of [15] and theorem 4 of [20].

In case $\lambda \neq 0$ the $n-2$ invariants of $\mathfrak{J}_\lambda^{n-1}$ are found by the normalization procedure applied to the lifted invariants $\widehat{\mathcal{I}}_1, \ldots, \widehat{\mathcal{I}}_{n-1}$. We solve $\widehat{\mathcal{I}}_2 = 0$ with respect to the parameter θ_n. Substitution of the obtained expression $\theta_n = -x_2/x_1$ into $\widehat{\mathcal{I}}_1$ and $\widehat{\mathcal{I}}_k/\widehat{\mathcal{I}}_1$, $k = 3, \ldots, n-1$, results in a basis of $\mathrm{Inv}(\mathfrak{J}_\lambda^{n-1})$:

$$\zeta_1 = e_1 \exp\left(-\lambda \frac{e_2}{e_1}\right), \qquad \zeta_k = \frac{\xi_k}{\xi_1^{k-1}}, \qquad k = 3, \ldots, n-1,$$

where ξ_k, $k = 1, 3, \ldots, n-1$, are defined by (2).

This set of invariants completely coincides with the one determined in lemma 2 of [15]. We only use exponential function instead of the logarithmic one in the expression of the first invariant. Let us emphasize that any basis of $\mathrm{Inv}(\mathfrak{J}_\lambda^{n-1})$ contains at least one transcendental invariant. The other basis invariants can be chosen rational.

The real version $\mathfrak{J}_{(\mu,\nu)}^{n-1}$ of the complex algebra $\mathfrak{J}_{\lambda\lambda^*}^{rr}$, where $n = 2r+1, r \in \mathbb{N}, \mu = \operatorname{Re}\lambda, \nu = \operatorname{Im}\lambda \neq 0$, has the non-zero commutation relations

$$[e_1, e_n] = \mu e_1 - \nu e_2, \qquad [e_2, e_n] = \nu e_1 + \mu e_2,$$
$$[e_{2k-1}, e_n] = \mu e_{2k-1} - \nu e_{2k} + e_{2k-3}, \qquad [e_{2k}, e_n] = \nu e_{2k-1} + \mu e_{2k} + e_{2k-2}, \qquad k = 2, \ldots, r.$$

A complete tuple $\widehat{\mathcal{I}}$ of lifted invariants has the form

$$\widehat{\mathcal{I}}_{2k-1} = \mathrm{e}^{\mu\theta_n}(\mathcal{I}_{2k-1}\cos\nu\theta_n - \mathcal{I}_{2k}\sin\nu\theta_n), \qquad \widehat{\mathcal{I}}_{2k} = \mathrm{e}^{\mu\theta_n}(\mathcal{I}_{2k-1}\sin\nu\theta_n + \mathcal{I}_{2k}\cos\nu\theta_n),$$
$$\widehat{\mathcal{I}}_n = \sum_{j=1}^{r}(\theta_{2j-1}(\mu x_{2j-1} - \nu x_{2j}) + \theta_{2j}(\nu x_{2j-1} + \mu x_{2j})) + \sum_{j=1}^{r-1}(\theta_{2j+1}x_{2j-1} + \theta_{2j+2}x_{2j}) + x_n,$$

where $k = 1, \ldots, r$,

$$\mathcal{I}_{2k-1} = \sum_{j=1}^{k} \frac{\theta_n^{k-j}}{(k-j)!} x_{2j-1}, \qquad \mathcal{I}_{2k} = \sum_{j=1}^{k} \frac{\theta_n^{k-j}}{(k-j)!} x_{2j}.$$

The normalization procedure is conveniently applied to the following combinations of the lifted invariants: $\widehat{\mathcal{I}}_{2k-1}, \widehat{\mathcal{I}}_{2k}$, $k = 1, \ldots, r$:

$$\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2 = \left(x_1^2 + x_2^2\right) \mathrm{e}^{2\mu\theta_n}, \qquad \arctan \frac{\widehat{\mathcal{I}}_2}{\widehat{\mathcal{I}}_1} = \arctan \frac{x_2}{x_1} + \nu\theta_n,$$
$$\frac{\widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_3 + \widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_4}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_1x_3 + x_2x_4}{x_1^2 + x_2^2} + \theta_n, \quad \frac{\widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_3 - \widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_4}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_2x_3 - x_1x_4}{x_1^2 + x_2^2},$$
$$\frac{\widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_{2k-1} + \widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_{2k}}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_1\mathcal{I}_{2k-1} + x_2\mathcal{I}_{2k}}{x_1^2 + x_2^2}, \quad \frac{\widehat{\mathcal{I}}_2\widehat{\mathcal{I}}_{2k-1} - \widehat{\mathcal{I}}_1\widehat{\mathcal{I}}_{2k}}{\widehat{\mathcal{I}}_1^2 + \widehat{\mathcal{I}}_2^2} = \frac{x_2\mathcal{I}_{2k-1} - x_1\mathcal{I}_{2k}}{x_1^2 + x_2^2}, \quad k = 3, \ldots, r.$$

We use the condition that the third combination (or second one if $n = 3$) equals to 0 as a normalization equation on the parameter θ_n and then exclude θ_n from the other combinations. It gives the basis of $\mathrm{Inv}\big(\mathfrak{J}^{n-1}_{(\mu,\nu)}\big)$

$$\zeta_1 = \left(e_1^2 + e_2^2\right) \exp\left(-2\frac{\mu}{\nu} \arctan \frac{e_2}{e_1}\right),$$
$$\zeta_3 = \nu \frac{e_1e_3 + e_2e_4}{e_1^2 + e_2^2} - \arctan \frac{e_2}{e_1}, \qquad \zeta_4 = \frac{e_1e_4 - e_2e_3}{e_1^2 + e_2^2},$$
$$\zeta_{2k-1} = \frac{e_1\hat{\zeta}_{2k-1} + e_2\hat{\zeta}_{2k}}{e_1^2 + e_2^2}, \qquad \zeta_{2k} = \frac{e_2\hat{\zeta}_{2k-1} - e_1\hat{\zeta}_{2k}}{e_1^2 + e_2^2}, \qquad k = 3, \ldots, r,$$

where

$$\hat{\zeta}_{2k-1} = \sum_{j=1}^{k} \left(-\frac{e_1e_3 + e_2e_4}{e_1^2 + e_2^2}\right)^{k-j} \frac{e_{2j-1}}{(k-j)!}, \qquad \hat{\zeta}_{2k} = \sum_{j=1}^{k} \left(-\frac{e_1e_3 + e_2e_4}{e_1^2 + e_2^2}\right)^{k-j} \frac{e_{2j}}{(k-j)!}.$$

Therefore, $\mathfrak{J}^2_{(\mu,\nu)}$ has unique independent invariant ζ_1 which is necessarily transcendental. In case $n = 2r + 1 \geqslant 5$ any basis of $\mathrm{Inv}\big(\mathfrak{J}^{n-1}_{(\mu,\nu)}\big)$ contains at least two transcendental invariants; the other $n - 4$ basis invariants can be chosen rational. A quite optimal basis with minimal number of transcendental invariants is formed by ζ_k, $k = 1, 3, \ldots, n - 1$.

6.2. General case

The inner automorphisms of $\mathfrak{J}^{r_1\cdots r_s}_{\lambda_1\cdots\lambda_s}$ are described by the triangular matrix

$$B(\theta) = \left(\exp\left(\theta_n J^{r_1}_{\lambda_1}\right) \oplus \cdots \oplus \exp\left(\theta_n J^{r_s}_{\lambda_s}\right)\right) \overset{C}{+} E^1, \qquad C = \left(C^{r_1}_{\lambda_1}, \ldots, C^{r_s}_{\lambda_s}\right)^{\mathrm{T}},$$
$$C^{r_j}_{\lambda_j} = \left(\lambda_j\theta_{\rho_j+1} + \theta_{\rho_j+2}, \ldots, \lambda_j\theta_{\rho_j+r_j-1} + \theta_{\rho_j+r_j}, \lambda_j\theta_{\rho_j+r_j}\right), \qquad j = 1, \ldots, s,$$
$$\rho_1 = 0, \qquad \rho_j = r_1 + \cdots + r_{j-1}, \qquad j = 2, \ldots, s.$$

The corresponding complete tuple $\widehat{\mathcal{I}} = \check{x} \cdot B(\theta)$ of lifted invariants has the form

$$\widehat{\mathcal{I}}_{\rho_j+q} = \mathrm{e}^{\lambda_j\theta_n} \sum_{p=1}^{q} \frac{1}{(q-p)!} \theta_n^{q-p} x_{\rho_j+p}, \qquad j = 1, \ldots, s, \qquad q = 1, \ldots, r_j,$$
$$\widehat{\mathcal{I}}_n = \sum_{j=1}^{s} \left(\lambda_j \sum_{q=1}^{r_j} \theta_{\rho_j+q} x_{\rho_j+q} + \sum_{q=1}^{r_j-1} \theta_{\rho_j+q+1} x_{\rho_j+q}\right) + x_n.$$

This tuple is obviously modified in the real case with complex eigenvalues.

The $n - 2$ invariants are found by the normalization procedure applied to the lifted invariants $\widehat{\mathcal{I}}_1, \ldots, \widehat{\mathcal{I}}_{n-1}$ in different ways. We can either use the same normalization equation for all Jordan blocks or normalize lifted invariants for each Jordan block separately and then simultaneously normalize some lifted invariants corresponding to different Jordan blocks. Intermediate variants are also possible. In any case, the procedure is reduced to choice of $n-2$ pairs from the lifted invariants $\widehat{\mathcal{I}}_1, \dots, \widehat{\mathcal{I}}_{n-1}$. The first term of each pair gives the left-hand side of the corresponding normalization equations. Substitution of the obtained value of the parameter θ_n into the second term of the pair results in an invariant of $\mathfrak{J}^{r_1\cdots r_s}_{\lambda_1\cdots\lambda_s}$. The constructed invariants form a basis of $\mathrm{Inv}\big(\mathfrak{J}^{r_1\cdots r_s}_{\lambda_1\cdots\lambda_s}\big)$ iff each from the lifted invariants $\widehat{\mathcal{I}}_1, \dots, \widehat{\mathcal{I}}_{n-1}$ falls within the $n-2$ chosen pairs at least once.

We use the strategy based on initial normalization of lifted invariants for each Jordan block separately. Then it is sufficient for us to describe the procedure for different kinds of pairs of Jordan blocks. Below we adduce short explanation on these pairs together with the optimally used pairs of lifted invariants and obtained invariants of the algebra; i, $j = 1, \dots, s$.

$J^{r_i}_{\lambda_i}$, $J^{r_j}_{\lambda_j}$:

$\lambda_i \neq 0, \lambda_j \neq 0$: $\quad \widehat{\mathcal{I}}_{\rho_i+1}, \widehat{\mathcal{I}}_{\rho_j+1}, \quad e^{-\lambda_j}_{\rho_i+1} e^{\lambda_i}_{\rho_j+1}$;

$r_i \geqslant 2, \lambda_i = 0, r_j \geqslant 2, \lambda_j = 0$: $\quad \widehat{\mathcal{I}}_{\rho_i+2}, \widehat{\mathcal{I}}_{\rho_j+2}, \quad e_{\rho_j+2}e_{\rho_i+1} - e_{\rho_i+2}e_{\rho_j+1}$;

$r_i \geqslant 2, \lambda_i \neq 0, r_j \geqslant 2, \lambda_j = 0$: $\quad \widehat{\mathcal{I}}_{\rho_i+2}, \widehat{\mathcal{I}}_{\rho_j+2}, \quad \dfrac{e_{\rho_j+2}}{e_{\rho_j+1}} - \dfrac{e_{\rho_i+2}}{e_{\rho_i+1}}$;

$r_i = 1, \lambda_i \neq 0, r_j \geqslant 2, \lambda_j = 0$: $\quad \widehat{\mathcal{I}}_{\rho_i+1}, \widehat{\mathcal{I}}_{\rho_j+2}, \quad e_{\rho_i+1} \exp\left(-\lambda_i \dfrac{e_{\rho_j+2}}{e_{\rho_j+1}}\right)$;

$J^{r_i}_{\lambda_i}$, $R^{2r_j}_{\mu_j\nu_j}$, $\lambda_i, \mu_j, \nu_j \in \mathbb{R}$, $\nu_j \neq 0$:

$r_i \geqslant 2, r_j \geqslant 2$: $\quad \widehat{\mathcal{I}}_{\rho_i+2}, \dfrac{\widehat{\mathcal{I}}_{\rho_j+2}\widehat{\mathcal{I}}_{\rho_j+3} - \widehat{\mathcal{I}}_{\rho_j+1}\widehat{\mathcal{I}}_{\rho_j+4}}{\widehat{\mathcal{I}}^2_{\rho_j+1} + \widehat{\mathcal{I}}^2_{\rho_j+2}}, \quad \dfrac{e_{\rho_j+1}e_{\rho_j+3} + e_{\rho_j+2}e_{\rho_j+4}}{e^2_{\rho_j+1} + e^2_{\rho_j+2}} - \dfrac{e_{\rho_i+2}}{e_{\rho_i+1}}$;

$r_i = 1$ or $r_j = 1$: $\quad \widehat{\mathcal{I}}_{\rho_i+1}, \arctan \dfrac{\widehat{\mathcal{I}}_{\rho_j+2}}{\widehat{\mathcal{I}}_{\rho_j+1}}, \quad e_{\rho_i+1} \exp\left(-\dfrac{\lambda_i}{\nu_j} \arctan \dfrac{e_{\rho_j+2}}{e_{\rho_j+1}}\right)$;

$R^{2r_i}_{\mu_i\nu_i}$, $R^{2r_j}_{\mu_j\nu_j}$, $\mu_i, \nu_i, \mu_j, \nu_j \in \mathbb{R}$, $\nu_i\nu_j \neq 0$:

$r_i \geqslant 2, r_j \geqslant 2$: $\quad \dfrac{\widehat{\mathcal{I}}_{\rho_i+2}\widehat{\mathcal{I}}_{\rho_i+3} - \widehat{\mathcal{I}}_{\rho_i+1}\widehat{\mathcal{I}}_{\rho_i+4}}{\widehat{\mathcal{I}}^2_{\rho_i+1} + \widehat{\mathcal{I}}^2_{\rho_i+2}}, \dfrac{\widehat{\mathcal{I}}_{\rho_j+2}\widehat{\mathcal{I}}_{\rho_j+3} - \widehat{\mathcal{I}}_{\rho_j+1}\widehat{\mathcal{I}}_{\rho_j+4}}{\widehat{\mathcal{I}}^2_{\rho_j+1} + \widehat{\mathcal{I}}^2_{\rho_j+2}}$,

$\dfrac{e_{\rho_j+1}e_{\rho_j+3} + e_{\rho_j+2}e_{\rho_j+4}}{e^2_{\rho_j+1} + e^2_{\rho_j+2}} - \dfrac{e_{\rho_i+1}e_{\rho_i+3} + e_{\rho_i+2}e_{\rho_i+4}}{e^2_{\rho_i+1} + e^2_{\rho_i+2}}$;

$r_i = 1$ or $r_j = 1$: $\quad \arctan \dfrac{\widehat{\mathcal{I}}_{\rho_i+2}}{\widehat{\mathcal{I}}_{\rho_i+1}}, \quad \arctan \dfrac{\widehat{\mathcal{I}}_{\rho_j+2}}{\widehat{\mathcal{I}}_{\rho_j+1}}, \quad \nu_i \arctan \dfrac{e_{\rho_j+2}}{e_{\rho_j+1}} - \nu_j \arctan \dfrac{e_{\rho_i+2}}{e_{\rho_i+1}}$.

The marginal case is $r_1 = \cdots = r_s = 1$, i.e. all Jordan blocks are one-dimensional. Let us recall that λ_k is assumed non-zero if $r_k = 1$. A complete set of generalized Casimir operators is formed by $e_1^{-\lambda_j} e_j^{\lambda_1}$, $j = 2, \dots, n-1$. In case $\lambda_j/\lambda_1 \in \mathbb{Q}$, the invariants can be made rational with raising to a power. If additionally λ_j/λ_1 have the same sign, $\mathrm{Inv}\big(\mathfrak{J}^{r_1\cdots r_s}_{\lambda_1\cdots\lambda_s}\big)$ has a polynomial basis, i.e. a basis consisting of usual Casimir operators.

Therefore, $\mathrm{Inv}\big(\mathfrak{J}^{r_1\cdots r_s}_{\lambda_1\cdots\lambda_s}\big)$ has a polynomial basis only in two cases

(1) $\lambda_1 = \cdots = \lambda_s = 0$, i.e. the algebra is nilpotent;

(2) $s = n-1 > 2$, $r_j = 1$, λ_j/λ_1 are rational and have the same sign, $j = 2, \dots, n-1$.

7. Solvable Lie algebras with nilradical $\mathfrak{J}_0^{n-1}$

Let us pass to complex indecomposable solvable Lie algebras with the nilradicals isomorphic to $\mathfrak{J}_0^{n-1}$, $n = 4, 5, \ldots$. All possible types of such algebras are described in theorems 1–3 of [20]. Their dimensions can be equal to $n + 1$ or $n + 2$. Below we adduce only the non-zero commutation relations, excluding ones between basis elements of the nilradicals:

$$[e_k, e_n] = e_{k-1}, \qquad k = 2, \ldots, n-1.$$

There exist three inequivalent classes of such algebras of dimension $n+1$. The first series $\mathfrak{s}_{1,n+1}$ is formed by Lie algebras $\mathfrak{s}_{1,n+1}^{\alpha\beta}$ with the additional non-zero commutation relations

$$[e_k, e_{n+1}] = \gamma_k e_k, \qquad k = 1, \ldots, n-1, \qquad [e_n, e_{n+1}] = \alpha e_n.$$

where $\gamma_k := (n-k-1)\alpha + \beta$. Due to scale transformations of e_{n+1} the parameter tuple (α, β) can be normalized to belong to the set $\{(1, \beta), (0, 1)\}$. We assume below that the parameters take only the normalized values. Then any algebras in the series $\mathfrak{s}_{1,n+1}$ are inequivalent to each other. For the values $(\alpha, \beta) \in \{(1, 0), (1, 2-n), (0, 1)\}$ the corresponding algebras have some singular properties.

The second class consists of the unique algebra $\mathfrak{s}_{2,n+1}$:

$$[e_k, e_{n+1}] = \gamma_k e_k, \qquad k = 1, \ldots, n-1, \qquad [e_n, e_{n+1}] = e_n + e_{n+1},$$

where $\gamma_k := n - k$.

The Lie algebra $\mathfrak{s}_{3,n+1}^{a_3,\ldots,a_{n-1}}$ from the latter $(n-3)$-parametric series $\mathfrak{s}_{3,n+1}$ is determined by the commutation relations

$$[e_k, e_{n+1}] = e_k + \sum_{i=1}^{k-2} a_{k-i+1} e_i, \qquad k = 1, \ldots, n-1,$$

where $a_j \in \mathbb{C}$, $j = 3, \ldots, n-1$, and $a_j \neq 0$ for some values of j.

The unique $(n+2)$-dimensional algebra $\mathfrak{s}_{4,n+2}$ of such type has the additional non-zero commutation relations

$$[e_k, e_{n+1}] = \gamma_k e_k, \qquad [e_n, e_{n+1}] = e_n, \qquad [e_k, e_{n+2}] = e_k, \qquad k = 1, \ldots, n-1,$$

where $\gamma_k := n - k - 1$.

The matrices determining the inner automorphisms of the above algebras are conveniently presented in the form $B(\theta) = B_1 B_2 B_3$, where

$$B_1 = \exp\left(\theta_1 \hat{\mathrm{ad}}_{e_1}\right) \cdots \exp\left(\theta_{n-1} \hat{\mathrm{ad}}_{e_{n-1}}\right), \quad B_2 = \exp\left(-\theta_n \hat{\mathrm{ad}}_{e_n}\right), \quad B_3 = \exp\left(-\theta_{n+1} \hat{\mathrm{ad}}_{e_{n+1}}\right),$$

excluding the $(n+2)$-dimensional case where $B_3 = \exp\left(-\theta_{n+1} \hat{\mathrm{ad}}_{e_{n+1}}\right) \exp\left(-\theta_{n+2} \hat{\mathrm{ad}}_{e_{n+2}}\right)$. The matrices B_1, B_2 and B_3 are written in a block form corresponding to partition of a basis of the algebra under consideration to the basis $e_1, \ldots, e_{n-1}$ of the maximal Abelian ideal and a complementary part.

For the algebra $\mathfrak{s}_{1,n+1}^{\alpha\beta}$

$$B_1 = E^{n-1} \overset{C}{+} E^2,$$

$$B_2 = \exp\left(\theta_n J_0^{n-1}\right) \oplus \begin{pmatrix} 1 & -\alpha\theta_n \\ 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} \theta_2 & \gamma_1\theta_1 \\ \theta_3 & \gamma_2\theta_2 \\ \cdots & \cdots \\ \theta_{n-1} & \gamma_{n-2}\theta_{n-2} \\ 0 & \gamma_{n-1}\theta_{n-1} \end{pmatrix},$$

$$B_3 = \mathrm{diag}(\mathrm{e}^{\gamma_1\theta_{n+1}}, \ldots, \mathrm{e}^{\gamma_{n-1}\theta_{n+1}}) \oplus \mathrm{diag}(\mathrm{e}^{\alpha\theta_{n+1}}, 1).$$

Therefore, the corresponding complete tuple $\widehat{\mathcal{I}} = \check{x} \cdot B(\theta)$ of lifted invariants has the form

$$\widehat{\mathcal{I}}_k = \mathrm{e}^{\gamma_k\theta_{n+1}}\mathcal{I}_k, \qquad k = 1,\dots,n-1,$$
$$\widehat{\mathcal{I}}_n = \mathrm{e}^{\alpha\theta_{n+1}}\mathcal{I}_n, \qquad \widehat{\mathcal{I}}_{n+1} = -\alpha\theta_n\mathcal{I}_n + \mathcal{I}_{n+1}.$$

Here $\mathcal{I}_1,\dots,\mathcal{I}_n$ are defined by (1), and

$$\mathcal{I}_{n+1} := \sum_{j=1}^{n-1}\gamma_j\theta_j x_j + x_{n+1}.$$

For the algebra $\mathfrak{s}_{2,n+1}$ only the matrices B_2 and B_3 and the lifted invariants $\widehat{\mathcal{I}}_n$ and $\widehat{\mathcal{I}}_{n+1}$ differ from those of the previous algebras:

$$B_2 = \exp\left(\theta_n J_0^{n-1}\right) \oplus \begin{pmatrix} 1 & \mathrm{e}^{-\theta_n}-1 \\ 0 & \mathrm{e}^{-\theta_n} \end{pmatrix},$$

$$B_3 = \mathrm{diag}(\mathrm{e}^{\gamma_1\theta_{n+1}},\dots,\mathrm{e}^{\gamma_{n-1}\theta_{n+1}}) \oplus \begin{pmatrix} \mathrm{e}^{\theta_{n+1}} & 0 \\ \mathrm{e}^{\theta_{n+1}}-1 & 1 \end{pmatrix},$$

$$\widehat{\mathcal{I}}_n = (\mathrm{e}^{\theta_{n+1}-\theta_n} - \mathrm{e}^{-\theta_n}+1)\mathcal{I}_n + \mathrm{e}^{-\theta_n}(\mathrm{e}^{\theta_{n+1}}-1)\mathcal{I}_{n+1}, \qquad \widehat{\mathcal{I}}_{n+1} = (\mathrm{e}^{-\theta_n}-1)\mathcal{I}_n + \mathrm{e}^{-\theta_n}\mathcal{I}_{n+1}.$$

All the $(n+1)$-dimensional algebras under consideration have exactly $n-3$ independent invariants which can be found by the normalization procedure applied to the lifted invariants $\widehat{\mathcal{I}}_1,\dots,\widehat{\mathcal{I}}_{n-1}$. Since the invariants of these algebras depend only on the element of the Abelian ideal the symmetrization procedure is trivial and can be omitted as a step of the algorithm.

For the algebras $\mathfrak{s}_{1,n+1}^{\alpha\beta}$, $(\alpha,\beta) \neq (1, 2-n)$ and $\mathfrak{s}_{2,n+1}$ we solve equations $\widehat{\mathcal{I}}_1 = 1$ and $\widehat{\mathcal{I}}_2 = 0$ with respect to the values $\mathrm{e}^{\theta_{n+1}}$ and θ_n. Substituting the obtained expressions $\mathrm{e}^{\theta_{n+1}} = x_1^{-1/\gamma_1}$ and $\theta_n = -x_2/x_1$ into the other $\widehat{\mathcal{I}}$'s, we get the following complete set of generalized Casimir operators

$$\xi_1^{-(k-1)\frac{(n-3)\alpha+\beta}{(n-2)\alpha+\beta}}\xi_k, \qquad k = 3,\dots,n-1,$$

where ξ_k, $k = 1, 3,\dots,n-1$, are defined by (2). For the algebra $\mathfrak{s}_{2,n+1}$ the value $\alpha=\beta=1$ should be taken.

The algebra $\mathfrak{s}_{1,n+1}^{1,2-n}$ is singular with respect to the normalization procedure and will be studied separately. In this case $\gamma_1 = 0$ and $\widehat{\mathcal{I}}_1 = x_1$ hence the basis element e_1 generating the centre of $\mathfrak{s}_{1,n+1}^{1,2-n}$ is one of the invariants. We obtain the expressions for θ_n and $\mathrm{e}^{\theta_{n+1}}$ from the system $\widehat{\mathcal{I}}_2 = 0, \widehat{\mathcal{I}}_3 = 1$ and substitute them into the other $\widehat{\mathcal{I}}$'s. Additionally we use the possibility of multiplication of invariants by powers of the invariant x_1. The resulting complete set of generalized Casimir operators is formed by

$$\xi_1, \qquad \frac{\xi_k^2}{\xi_3^{k-1}}, \qquad k = 4,\dots,n-1.$$

Calculations for the Lie algebra $\mathfrak{s}_{3,n+1}^{a_3,\dots,a_{n-1}}$ are analogous but more complicated:

$$B_1 = E^{n-1} \overset{C}{+} E^2, \qquad C = \begin{pmatrix} \theta_2 & \theta_1 + a_3\theta_3 + a_4\theta_4 + \dots + a_{n-1}\theta_{n-1} \\ \theta_3 & \theta_2 + a_3\theta_4 + a_4\theta_5 + \dots + a_{n-2}\theta_{n-1} \\ \dots & \dots \\ \theta_{n-3} & \theta_{n-3} + a_3\theta_{n-2} \\ \theta_{n-1} & \theta_{n-2} \\ 0 & \theta_{n-1} \end{pmatrix},$$

$$B_2 = \exp\left(\theta_n J_0^{n-1}\right) \oplus E^2,$$

$$B_3 = \mathrm{e}^{\theta_{n+1}}\left(E^{n-1} + \sum_{m=2}^{n-2}\left(J_0^{n-1}\right)^m \sum_{i=1}^{[\frac{m}{2}]}\frac{b_{mi}}{i!}\theta_{n+1}^i\right) \oplus E^2, \qquad b_{mi} = \sum_{\substack{3\leqslant s_1,\dots,s_i\leqslant n-1 \\ s_1+\dots+s_i=m+i}} a_{s_1}\cdots a_{s_i}.$$

The corresponding complete tuple $\widehat{\mathcal{I}} = \check{x} \cdot B(\theta)$ of lifted invariants has the form

$$\widehat{\mathcal{I}}_k = \mathrm{e}^{\theta_{n+1}}\left(\mathcal{I}_k + \sum_{m=2}^{k-1} \mathcal{I}_{k-m} \sum_{i=1}^{[\frac{m}{2}]} \frac{b_{mi}}{i!}\theta_{n+1}^i\right), \qquad k = 1, \ldots, n-1,$$

$$\widehat{\mathcal{I}}_n = \mathcal{I}_n, \qquad \widehat{\mathcal{I}}_{n+1} = \mathcal{I}_{n+1} + \sum_{k=1}^{n-1} x_k \sum_{i=1}^{n-k-2} \theta_{k+1+i} a_{i+2}.$$

The Lie algebra $\mathfrak{s}_{3,n+1}^{a_3,\ldots,a_{n-1}}$ has $n-3$ invariants for any values of the parameters. Applying the normalization procedure to $\widehat{\mathcal{I}}$, we solve the system $\mathcal{I}_1 = 1, \mathcal{I}_2 = 0$ with respect to θ_n and θ_{n+1}. Substitution of the obtained expressions $\theta_n = -x_2/x_1$ and $\theta_{n+1} = -\ln x_1$ into the other $\mathcal{I}$'s gives the following complete set of generalized Casimir operators:

$$\xi_1^{-k+1}\xi_k + \sum_{m=2}^{k-1} \xi_1^{-k+m+1}\xi_{k-m} \sum_{i=1}^{[\frac{m}{2}]} \frac{b_{mi}}{i!}(-\ln \xi_1)^i, \qquad k = 3, \ldots, n-1,$$

where ξ_k, $k = 1, 3, \ldots, n-1$, are defined by (2).

For the algebra $\mathfrak{s}_{4,n+2}$

$$B_1 = E^{n-1} \overset{C}{+} E^3, \qquad C = \begin{pmatrix} \theta_2 & \gamma_1\theta_1 & \theta_1 \\ \theta_3 & \gamma_2\theta_2 & \theta_2 \\ \cdots & \cdots & \cdots \\ \theta_{n-1} & \gamma_{n-2}\theta_{n-2} & \theta_{n-2} \\ 0 & \gamma_{n-1}\theta_{n-1} & \theta_{n-1} \end{pmatrix}, \qquad \gamma_k := n-k-1,$$

$$B_2 = \exp\left(\theta_n J_0^{n-1}\right) \oplus \begin{pmatrix} 1 & -\theta_n & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_3 = \mathrm{e}^{\theta_{n+2}}\mathrm{diag}(\mathrm{e}^{\gamma_1\theta_{n+1}}, \ldots, \mathrm{e}^{\gamma_{n-1}\theta_{n+1}}) \oplus \mathrm{diag}(\mathrm{e}^{\alpha\theta_{n+1}}, 1, 1),$$

i.e. the tuple $\widehat{\mathcal{I}} = \check{x} \cdot B(\theta)$ of lifted invariants has the form

$$\widehat{\mathcal{I}}_k = \mathrm{e}^{\gamma_k\theta_{n+1}+\theta_{n+2}}\mathcal{I}_k, \qquad k = 1, \ldots, n-1,$$

$$\widehat{\mathcal{I}}_n = \mathrm{e}^{\theta_{n+1}}\mathcal{I}_n, \qquad \widehat{\mathcal{I}}_{n+1} = \mathcal{I}_{n+1} - \theta_n\mathcal{I}_n, \qquad \widehat{\mathcal{I}}_{n+2} = \mathcal{I}_{n+2} := \sum_{j=1}^{n-1} \theta_j x_j + x_{n+2}.$$

The $n-4$ invariants of $\mathfrak{s}_{4,n+2}$ are found by the normalization procedure applied to the lifted invariants $\widehat{\mathcal{I}}_1, \ldots, \widehat{\mathcal{I}}_{n-1}$. We solve the system $\widehat{\mathcal{I}}_1 = 1, \widehat{\mathcal{I}}_2 = 0, \widehat{\mathcal{I}}_3 = 1$ with respect to the parameters θ_n, θ_{n+1} and θ_{n+2} and then exclude them from the other $\widehat{\mathcal{I}}$'s. As a result, we obtain a complete set of invariants of $\mathfrak{s}_{4,n+2}$:

$$\frac{\xi_k^2}{\xi_3^{k-1}}, \qquad k = 4, \ldots, n-1,$$

where ξ_k, $k = 3, \ldots, n-1$, are defined by (2).

The sets of generalized Casimir invariants for the Lie algebras with the nilradicals isomorphic to $\mathfrak{J}_0^{n-1}$, which are constructed in this section, coincide with the ones determined in theorems 5 and 6 of [20].

8. Nilpotent algebra of strictly upper triangle matrices

Consider the nilpotent Lie algebra $\mathfrak{t}_0(n)$ isomorphic to the one of strictly upper triangle $n \times n$ matrices. $\mathfrak{t}_0(n)$ has dimension $n(n-1)/2$. It is the Lie algebra of the Lie group $T_0(n)$ of upper unipotent $n \times n$ matrices, i.e. upper triangular matrices with the unities on the diagonal.

Its basis elements are convenient to enumerate with the 'increasing' pair of indices similarly to the canonical basis $\{E^n_{ij}, i < j\}$ of the isomorphic matrix algebra. Thus, the basis elements $e_{ij} \sim E^n_{ij}$, $i < j$, satisfy the commutation relations

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj},$$

where δ_{ij} is the Kronecker delta.

Hereafter the indices i, j, k and l run at most from 1 to n. Only additional constraints on the indices are indicated.

Let e^*_{ji}, x_{ji} and y_{ij} denote the basis element and the coordinate function in the dual space $\mathfrak{t}^*_0(n)$ and the coordinate function in $\mathfrak{t}_0(n)$, which correspond to the basis element e_{ij}, $i < j$. We complete the sets of x_{ji} and y_{ij} to the matrices X and Y with zeros. Hence X is a strictly lower triangle matrix and Y is a strictly upper triangle one.

Lemma 1. *A complete set of independent lifted invariants of* $\mathrm{Ad}^*_{T_0(n)}$ *is exhausted by the expressions*

$$\mathcal{I}_{ij} = x_{ij} + \sum_{i<i'} b_{ii'} x_{i'j} + \sum_{j'<j} \widehat{b}_{j'j} x_{ij'} + \sum_{i<i',\, j'<j} b_{ii'} \widehat{b}_{j'j} x_{i'j'}, \qquad j < i,$$

where $B = (b_{ij})$ *is an arbitrary matrix from* $T_0(n)$*;* $\widehat{B} = (\widehat{b}_{ij})$ *is the inverse matrix of* B*.*

Proof. The adjoint action of $B \in T_0(n)$ on the matrix Y is $\mathrm{Ad}_B Y = BYB^{-1}$, i.e.

$$\mathrm{Ad}_B \sum_{i<j} y_{ij} e_{ij} = \sum_{i<j} (BYB^{-1})_{ij} e_{ij} = \sum_{i\leqslant i'<j'\leqslant j} b_{ii'} y_{i'j'} \widehat{b}_{j'j} e_{ij}.$$

After changing $e_{ij} \to x_{ji}$, $y_{ij} \to e^*_{ji}$, $b_{ij} \leftrightarrow \widehat{b}_{ij}$ in the latter equality, we obtain the representation for the coadjoint action of B

$$\mathrm{Ad}^*_B \sum_{i<j} x_{ji} e^*_{ji} = \sum_{i\leqslant i'<j'\leqslant j} b_{j'j} x_{ji} \widehat{b}_{ii'} e^*_{j'i'} = \sum_{i'<j'} (BXB^{-1})_{j'i'} e^*_{j'i'}.$$

Therefore, the elements $\mathcal{I}_{ij}$, $j < i$, of the matrix

$$\mathcal{I} = BXB^{-1}, \qquad B \in T_0(n),$$

form a complete set of independent lifted invariants of $\mathrm{Ad}^*_{T_0(n)}$. □

Note 1. The centre of the group $T_0(n)$ is $Z(T_0(n)) = \{E^n + b_{1n} E^n_{1n}, b_{1n} \in \mathbb{F}\}$. The inner automorphism group of $\mathfrak{t}_0(n)$ is isomorphic to the factor-group $T_0(n)/Z(T_0(n))$ and hence its dimension is $\frac{1}{2}n(n-1) - 1$. The parameter b_{1n} in the above representation of lifted invariants is inessential.

Below $A^{i_1,i_2}_{j_1,j_2}$, where $i_1 \leqslant i_2$, $j_1 \leqslant j_2$, denotes the submatrix $(a_{ij})^{i=i_1,\ldots,i_2}_{j=j_1,\ldots,j_2}$ of a matrix $A = (a_{ij})$.

Lemma 2. *A set of independent invariants of* $\mathrm{Ad}^*_{T_0(n)}$ *is given by the expressions*

$$\det X^{n-k+1,n}_{1,k}, \qquad k = 1, \ldots, \left[\frac{n}{2}\right].$$

Proof. The derived formula for $\mathcal{I}$ and (triangle) structure of the matrices B and X imply that

$$\mathcal{I}_{1,k}^{n-k+1,n} = B_{n-k+1,n}^{n-k+1,n} X_{1,k}^{n-k+1,n} \widehat{B}_{1,k}^{1,k}, \qquad k = 1, \dots, \left[\frac{n}{2}\right].$$

(These submatrices have size $k \times k$ and lie in the left lower angle of $\mathcal{I}$, in the right lower angle of B, in the left lower angle of X and in the left upper angle of $\widehat{B}$ correspondingly.) Then

$$\det \mathcal{I}_{1,k}^{n-k+1,n} = \det X_{1,k}^{n-k+1,n}, \qquad k = 1, \dots, \left[\frac{n}{2}\right],$$

since $\det B_{n-k+1,n}^{n-k+1,n} = \det \widehat{B}_{1,k}^{1,k} = 1$, i.e. $\det X_{1,k}^{n-k+1,n}$ are invariants of $\mathrm{Ad}^*_{T_0(n)}$ in view of the definition of invariant. Functional independence of these invariants is obvious. □

Lemma 3. *The number of independent invariants of* $\mathrm{Ad}^*_{T_0(n)}$ *is not greater than* $\left[\frac{n}{2}\right]$.

Proof. Since $\det B = 1$, $\widehat{b}_{ij}$ for $i < j$ is algebraic complement to b_{ij} and then

$$\widehat{b}_{ij} = (-1)^{i+j} \det B_{i+1,j}^{i,j-1} = -b_{ij} + \cdots,$$

where the rest terms are polynomial in $b_{i'j'}$, $i' = i, \dots, j-1$, $j' = i+1, \dots, j$, $(i', j') \neq (i, j)$, $i' < j'$. These elements $b_{i'j'}$ are over the leading diagonal of B and not to the right of and not over b_{ij}.

We order and enumerate the lifted invariants $\mathcal{I}_{ij}$, $j < i$, $i + j \neq n+1$, in the following way:

$\mathcal{I}_{n-k+1,j}$, $j = 1, \dots, \min(k-1, n-k)$, $\qquad \mathcal{I}_{ik}$, $i = \max(k+1, n-k+2), \dots, n$,
$k = 2, \dots, n-1$,

and then enumerate them. The numeration matrix will look as

$$\begin{pmatrix}
\times & & & & & & & & \\
m_n-1 & \times & & & & & & & \\
m_n-5 & m_n-4 & \times & & & & & & \\
m_n-11 & m_n-10 & m_n-9 & \times & & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & & & & \\
7 & 8 & 9 & \times & \dots & \times & & & \\
3 & 4 & \times & 10 & \dots & m_n-6 & \times & & \\
1 & \times & 5 & 11 & \dots & m_n-7 & m_n-2 & \times & \\
\times & 2 & 6 & 12 & \dots & m_n-8 & m_n-3 & m_n & \times
\end{pmatrix},$$

where

$$m_n = \frac{n(n-1)}{2} - \left[\frac{n}{2}\right].$$

The obtained tuple of lifted invariants is denoted by $\mathcal{I}_{\prec}$.

In similar way we order and enumerate the parameters b_{ij}, $i < j$, $i + j \neq n+1$:

$b_{n-k+1,j}$, $j = \max(k+1, n-k+2), \dots, n$, $\qquad b_{ik}$, $i = 1, \dots, \min(k-1, n-k)$,
$k = 2, \dots, n-1$.

The corresponding numeration matrix is obtained from the previous numeration matrix with transposition and inversion of order of choosing pairs from rows and columns. The obtain tuple of parameters is denoted by $b_{\succ}$.

In view of the representation of lifted invariants, the Jacobian matrix $\partial \mathcal{I}_{\prec} / \partial b_{\succ}$ is block lower triangle of dimension m_n with the non-singular blocks

$$X_{1,k}^{n-k+1,n}, \quad \left(X_{1,k}^{n-k+1,n}\right)^{\mathrm{T}}, \quad k = 1, \dots, \left[\frac{n}{2}\right],$$

$$X_{1,k}^{n-k+1,n}, \quad \left(X_{1,k}^{n-k+1,n}\right)^{\mathrm{T}}, \quad k = \left[\frac{n}{2}\right], \dots, 1,$$

on the leading diagonal. Therefore, this matrix is non-singular and the rank of the complete Jacobian matrix of derivatives of the lifted invariants with respect to the parameters is not less than m_n. Then the number of independent invariants of $\mathfrak{t}_0(n)$ is

$$N_{\mathfrak{t}_0(n)} = \dim \mathfrak{t}_0(n) - \operatorname{rank} \mathfrak{t}_0(n) \leqslant \frac{n(n-1)}{2} - m_n = \left[\frac{n}{2}\right]. \qquad \square$$

Theorem 1. *A basis of* $\operatorname{Inv}(\mathfrak{t}_0(n))$ *is formed by the Casimir operators*

$$\det(e_{ij})_{j=n-k+1,n}^{i=1,\ldots,k}, \qquad k = 1, \ldots, \left[\frac{n}{2}\right].$$

Proof. Lemmas 2 and 3 immediately result in that the expressions from lemma 2 form a basis of $\operatorname{Inv}\big(\mathrm{Ad}^*_{T_0(n)}\big)$. Since the basis elements corresponding the coordinate functions in these expressions commutate to each other, the symmetrization procedure is trivial. $\square$

The above basis of invariants was first obtained in a quite heuristic way in [22].

9. Concluding remarks

The algebraic algorithm for computing the invariants of Lie algebras by means of moving frames of [2], intended originally for Lie algebras of fixed relatively dimension, is shown to be an efficient method for computing invariant operators for families of solvable Lie algebras, which share the same structure of nilradicals, but are of general dimension $n < \infty$. Moreover, it is clear from the results in this paper that the method is neither limited to such Lie algebras nor to the problem of finding generalized Casimir operators.

There are two other very different challenging problems in Lie theory which we want to point out in the expectation that the moving frame method could be adapted to their solution.

Consider a pair of Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$ such that $\mathfrak{g} \supset \mathfrak{g}'$. The generalized Casimir operators, we are finding here, clearly stabilize $\mathfrak{g}'$ inside $\mathfrak{g}$. One may expect that there are other functions of elements of $\mathfrak{g}$ that commute with the subalgebra $\mathfrak{g}'$. What are they? and what is their basis? Among semisimple Lie algebras, the answer has been given for two cases. Namely $SU(3) \supset O(3)$ in [10], and $SU(4) \supset SU(2) \times SU(2)$ in [18]. In the first case there are two additional operators in the universal enveloping algebra of $SU(3)$ that commutate with the subalgebra (but do not commute among themselves!). In the second case, four additional operators were found, two and two commuting.

The generalized Casimirs can be interpreted as a basis for the trivial one-dimensional representation of $\mathfrak{g}$. The second problem is to describe basis operators for other representations of $\mathfrak{g}$ than the one-dimensional that, for example, for the adjoint representation of $\mathfrak{g}$. The answer to this question has been given for many semisimple Lie algebras and for their various representations. See [16] and references therein.

We hope that the developed approach will be effectively used in other areas of mathematics and physics where the problem of finding functional bases of invariants of Lie algebras is actual. This approach can be extended in a natural way to invariants of Lie superalgebras, Poisson–Lie algebras etc. Investigation of (generalized) Casimir invariants of such algebras is an important problem of theoretical and mathematical physics, in particular, of the theory of integrable and superintegrable systems.

In case of low-dimensional Lie algebras our method can be easy realized by means of symbolic calculation packages.

Acknowledgments

The work was partially supported by the National Science and Engineering Research Council of Canada, by MITACS. The research of RP was supported by Austrian Science Fund (FWF), Lise Meitner project M923-N13. VB is grateful for the hospitality of the Centre de Recherches Mathématiques, Université de Montréal.

References

[1] Ancochea J M, Campoamor-Stursberg R and Garcia Vergnolle L 2006 Solvable Lie algebras with naturally graded nilradicals and their invariants *J. Phys. A: Math. Gen.* **39** 1339–55 (*Preprint* math-ph/0511027)

[2] Boyko V, Patera J and Popovych R 2006 Computation of invariants of Lie algebras by means of moving frames *J. Phys. A: Math. Gen.* **39** 5749–62 (*Preprint* math-ph/0602046)

[3] Bremner M R, Moody R V and Patera J 1985 Tables of dominant weight multiplicities for representations of simple Lie algebras *Monographs and Textbooks in Pure and Applied Mathematics* vol 90 (New York: Dekker)

[4] Campoamor-Stursberg R 2005 Some remarks concerning the invariants of rank one solvable real Lie algebras *Algebr. Colloq.* **12** 497–518

[5] Campoamor-Stursberg R 2006 Application of the Gel'fand matrix method to the missing label problem in classical kinematical Lie algebras *SIGMA* **2** Paper 028, 11 pages (*Preprint* math-ph/0602065)

[6] Casimir H B G 1931 Über die Konstruktion einer zu den irreduzibelen Darstellungen halbeinfaqer kontinuierliqer Gruppen gehörigen Differentialgleiqung *Proc. R. Acad. Amsterdam* **34** 844–6

[7] Fels M and Olver P 1998 Moving coframes: I. A practical algorithm *Acta Appl. Math.* **51** 161–213

[8] Fels M and Olver P 1999 Moving coframes: II. Regularization and theoretical foundations *Acta Appl. Math.* **55** 127–208

[9] Jacobson N 1955 *Lie Algebras* (New York: Interscience)

[10] Judd B, Miller W, Patera J and Winternitz P 1974 Complete sets of commuting operators and $O(3)$ scalars in the enveloping algebra of $SU(3)$ *J. Math. Phys.* **15** 1787–99

[11] Mubarakzyanov G M 1963 On solvable Lie algebras *Izv. Vys. Ucheb. Zaved. Matematika* no 1 (32) 114–23 (in Russian)

[12] Mubarakzyanov G M 1963 Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element *Izv. Vys. Ucheb. Zaved. Matematika* no 4 (35) 104–16 (in Russian)

[13] Ndogmo J C 2000 Properties of the invariants of solvable Lie algebras *Canad. Math. Bull.* **43** 459–71

[14] Ndogmo J C and Winternitz P 1994 Solvable Lie algebras with Abelian nilradicals *J. Phys. A: Math. Gen.* **27** 405–23

[15] Ndogmo J C and Winternitz P 1994 Generalized Casimir operators of solvable Lie algebras with Abelian nilradicals *J. Phys. A: Math. Gen.* **27** 2787–800

[16] Patera J 2003 R T Sharp and generating functions in group representation theory *CRM Proc. Lect. Notes* **34** 159–63

[17] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 Invariants of real low dimension Lie algebras *J. Math. Phys.* **17** 986–94

[18] Quesne C 1976 $SU(2) \times SU(2)$ scalars in the enveloping algebra of $SU(4)$ *J. Math. Phys.* **17** 1452–67

[19] Rubin J L and Winternitz P 1993 Solvable Lie algebras with Heisenberg ideals *J. Phys. A: Math. Gen.* **26** 1123–38

[20] Snobl L and Winternitz P 2005 A class of solvable Lie algebras and their Casimir invariants *J. Phys. A: Math. Gen.* **38** 2687–700 (*Preprint* math-ph/0411023)

[21] Tremblay S and Winternitz P 1998 Solvable Lie algebras with triangular nilradicals *J. Phys. A: Math. Gen.* **31** 789–806

[22] Tremblay S and Winternitz P 2001 Invariants of the nilpotent and solvable triangular Lie algebras *J. Phys. A: Math. Gen.* **34** 9085–99